NORTH-HOLLAND

# A Norm Estimate for the ADI Method For Nonsymmetric Problems*

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ABSTRACT

We give a norm estimate for the alternating direction implicit method for nonsymmetric elliptic convection-diffusion problems on a rectangular domain. We estimate a certain form of the iteration matrix in terms of the coefficients of convective terms and the mesh size. The norm is shown to be asymptotically of the form $(1 - Ch)/(1 + Ch)$, where $C$ is the same constant as in the symmetric case. We also show that the optimal size of the parameter is the same as in the symmetric case. As a consequence, we conclude that the convergence behavior is as good as that of the symmetric case and does not deteriorate as the size of convective terms grows. Numerical experiment shows that our analysis is sharp. © 1997 Elsevier Science Inc.

## 1. INTRODUCTION

Since the introduction of the alternating direction implicit method by Peaceman and Rachford [10] for solving an elliptic boundary value problem for a model of oil reservoirs, there has been extensive research on theoretical aspects [6, 2, 3, 5] as well as applications to physical problems [1, 4, 7–9, 11]. But for some time the convergence was known only for the case when the splitting matrices commute. It is only recently that the convergence of the

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ADI method in the case of noncommuting splitting was shown [4]. But there the commuting part was used as a preconditioner to resolve the convection dominated problem, through eigenvalue analysis of the iteration matrix. Moreover, it was assumed that either $bh/2 < 1$ or $ch/2 < 1$, excluding the case when both are greater than 1. In this paper, we estimate the norm of the iteration matrix resulting from ADI formulation for a nonsymmetric problem using eigenvectors of Laplace operators and show that the norm is of the form $(1 - Ch)/(1 + Ch)$ even if both $bh/2, ch/2$ are greater than 1. We also show that the asymptotic convergence factor is as good as the symmetric case with the same optimal choice of the iteration parameter, which has not been shown before. As a consequence, the convergence behavior is not affected by the size of convective terms.

The rest of our paper is organized as follows: In Section 2, we give a basic ADI formulation for solving stationary convection-diffusion problems. We use a weighted norm to estimate the norms of the iteration matrices corresponding to the horizontal sweep and vertical sweep, thus obtaining an estimate of the complete iteration matrix of the ADI method. In Section 3, we give a numerical example. It is shown that the number of iterations grows linearly with the number of unknowns, which indicates the sharpness of our estimate.

## 2. PROBLEM FORMULATION

Consider a second order elliptic partial differential equation on a rectangular domain $\Omega$, which we take as the unit square for simplicity:

$$\begin{aligned} -\Delta u + bu_x + cu_y &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator and $b, c$ are constants. Applying a central difference scheme gives rise to a system of algebraic equation of the form, after multiplication by $h^2$,

$$Ax = f, \qquad A = H_P + V_Q, \quad H_P = H + P, \; V_Q = V + Q,$$

where $H, V$ are respectively horizontal and vertical operators corresponding to $-\Delta$, and $P, Q$ are horizontal and vertical operators corresponding to $bu_x$

and $cu_y$. Explicitly, we have

$$H = \begin{bmatrix} J & 0 & 0 & \cdots & 0 & 0 \\ 0 & J & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & J \end{bmatrix},$$

where

$$J = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix},$$

and

$$V = \begin{bmatrix} 2I & -I & 0 & \cdots & 0 & 0 \\ -I & 2I & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2I & -I \\ 0 & 0 & 0 & \cdots & -I & 2I \end{bmatrix}.$$

$P$, $Q$ are block matrices of the form

$$P = \begin{bmatrix} B & 0 & 0 & \cdots & 0 & 0 \\ 0 & B & 0 & \cdots & 0 & 0 \\ 0 & 0 & B & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & B \end{bmatrix},$$

where

$$B = \frac{h}{2}\begin{bmatrix} 0 & b & 0 & \cdots & 0 & 0 \\ -b & 0 & b & \cdots & 0 & 0 \\ 0 & -b & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b \\ 0 & 0 & 0 & \cdots & -b & 0 \end{bmatrix},$$

and

$$Q = \begin{bmatrix} 0 & C & 0 & \cdots & 0 & 0 \\ -C & 0 & C & \cdots & 0 & 0 \\ 0 & -C & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & C \\ 0 & 0 & 0 & \cdots & -C & 0 \end{bmatrix},$$

where

$$C = \frac{h}{2}\operatorname{diag}[c, c, \dots, c].$$

Now we consider an ADI or Peaceman-Rachford iteration in the following form:

$$x^{k+1} = \left(I + sV_Q\right)^{-1}\left(I + sH_P\right)^{-1}\left(I - sH_P\right)\left(I - sV_Q\right)x^k$$

$$+\left(I + sV_Q\right)^{-1}\left(I + sH_P\right)^{-1}2sAx, \tag{2.2}$$

$$x^{k+1} = G_s x^k + \tilde{f}. \tag{2.3}$$

Here $s$ is a certain positive number to be determined later. For Laplace equations on the rectangle domain, it is well known that the reciprocal of the geometric mean of two extreme eigenvalues is the optimal choice [2, 3]. Let

$$\tilde{G}_s = \left(I + sV_Q\right)G_s\left(I + sV_Q\right)^{-1}$$

$$= \left(I + sH_P\right)^{-1}\left(I - sH_P\right)\left(I - sV_Q\right)\left(I + sV_Q\right)^{-1}. \tag{2.4}$$

Let $S = I + sV_Q$. Then $\tilde{G}_s = SG_sS^{-1}$, and if we introduce a new norm $\|x\|_S$ defined by

$$\|x\|_S = (Sx, Sx)^{1/2},$$

we have

$$\begin{aligned}\|\tilde{G}_s\| &= \sup_{\|x\|=1} \|SG_s S^{-1}x\| \\ &= \sup_{\|Sy\|=1} \|SG_s y\| \\ &= \sup_{\|y\|_S=1} \|G_s y\|_S \\ &= \|G\|_S.\end{aligned}$$

Thus to estimate $\|G\|_S$, it suffices to estimate $\|(I+sH_P)^{-1}(I-sH_P)\|$ and $\|(I-sV_Q)(I+sV_Q)^{-1}\|$.

Since $H_P = \operatorname{diag}[J_B, \dots, J_B]$, where $J_B = J + B$ we see that $\|H_P\| = \|J_B\|$. Thus to estimate $\|(I+sH_P)^{-1}(I-sH_P)\|$, it suffices to estimate

$$\left\|(I+sJ_B)^{-1}(I-sJ_B)\right\|. \tag{2.5}$$

Let $h = 1/(n+1)$, and let $x_j = [\sin j\pi h, \dots, \sin nj\pi h]^T$ be the eigenvector of $J$ corresponding to the eigenvalue $\lambda_j = 2(1-\cos j\pi h)$.

LEMMA 2.1. *We have*

$$JBx_j - BJx_j = (bh \sin j\pi h)\left[1, 0, \dots, 0, (-1)^j\right] \tag{2.6}$$

*and*

$$B^2 x_j = -(b^2h^2 \sin^2 j\pi h)x_j + \left(\frac{b^2h^2}{2}\sin j\pi h\right)\left[1, 0, \dots, 0, (-1)^{j-1}\right]. \tag{2.7}$$

*Proof.* Since

$$\begin{aligned}Bx_j &= \frac{bh}{2}[\sin 2j\pi h, \sin 3j\pi h - \sin j\pi h, \dots, \sin(n-1)j\pi h] \\ &= (bh \sin j\pi h)[\cos j\pi h, \cos 2j\pi h, \dots, \cos nj\pi h],\end{aligned} \tag{2.8}$$

we have

$$\begin{aligned}
JBx_j &= (bh \sin j\pi h)[2\cos j\pi h - 2\cos 2j\pi h, 2\cos 2j\pi h - \cos j\pi h \\
&\qquad - \cos 3j\pi h, \dots, 2\cos nj\pi h - \cos(n-1)j\pi h] \\
&= (2bh \sin j\pi h)(1 - \cos j\pi h)[0, \cos 2j\pi h, \dots, \cos(n-1)j\pi h, 0] \\
&\quad + (bh \sin j\pi h)[2\cos j\pi h - \cos 2j\pi h, 0, \dots, 0, 2\cos nj\pi h \\
&\qquad - \cos(n-1)j\pi h] \qquad (2.9)
\end{aligned}$$

and

$$\begin{aligned}
BJx_j &= 2(1 - \cos j\pi h) Bx_j \\
&= (2bh \sin j\pi h)(1 - \cos j\pi h)[\cos j\pi h, \cos 2j\pi h, \dots, \cos nj\pi h]. \qquad (2.10)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
JBx_j - BJx_j &= (bh \sin j\pi h)[2\cos j\pi h - \cos 2j\pi h, 0, \dots, 0, 2\cos nj\pi h \\
&\qquad - \cos(n-1)j\pi h] \\
&\quad - 2bh \sin(j\pi h)(1 - \cos j\pi h)[\cos j\pi h, 0, \dots, 0, \cos nj\pi h] \\
&= (bh \sin j\pi h)\left[1, 0, \dots, 0, (-1)^j\right]
\end{aligned}$$

For (2.7), and we see from (2.8) that

$$\begin{aligned}
B^2 x_j &= (bh \sin j\pi h) B[\cos j\pi h, \cos 2j\pi h, \dots, \cos nj\pi h] \\
&= \left(\frac{b^2h^2}{2} \sin j\pi h\right)[\cos 2j\pi h, \cos 3j\pi h - \cos j\pi h, \dots, \\
&\qquad \cos nj\pi h - \cos(n-2)j\pi h, -\cos(n-1)j\pi h] \\
&= -(b^2h^2 \sin^2 j\pi h)[0, \sin 2j\pi h, \dots, \sin(n-1)j\pi h, 0] \\
&\quad + \left(\frac{h^2}{2} \sin j\pi h\right)[\cos 2j\pi h, 0, \dots, 0, -\cos(n-1)j\pi h]
\end{aligned}$$

$$= -(b^2h^2 \sin^2 j\pi h)x_j + (h^2 \sin^2 j\pi h)[\sin j\pi h, 0, \dots, 0, \sin nj\pi h]$$

$$+\left(\frac{b^2h^2}{2} \sin j\pi h\right)[\cos 2j\pi h, 0, \dots, 0, -\cos(n-1)j\pi h]$$

$$= -(b^2h^2 \sin^2 j\pi h)x_j + \left(\frac{h^2}{2} \sin j\pi h\right)\left[1, 0, \dots, 0, (-1)^{j-1}\right],$$

which completes the proof. ■

LEMMA 2.2. *We have*

$$x_i^T J_B^T J_B x_j = \delta_{ij}\left(\lambda_j^2 + b^2h^2 \sin^2 j\pi h\right)x_i^T x_j$$

$$+(bh \sin j\pi h)\left[1 + (-1)^{i+j-1}\right] \sin i\pi h$$

$$-\left(\frac{b^2h^2}{2} \sin j\pi h\right)\left[1 + (-1)^{i+j}\right] \sin i\pi h.$$

*Proof.* Note that

$$J_B^T J_B = J^2 + JB - BJ - B^2.$$

From Lemma 2.2, we have

$$J_B^T J_B x_j = \lambda_j^2 x_j + (bh \sin j\pi h)\left[1, 0, \dots, 0, (-1)^j\right] + \left(b^2h^2 \sin^2 j\pi h\right)x_j$$

$$-\left(\frac{b^2h^2}{2} \sin j\pi h\right)\left[1, 0, \dots, 0, (-1)^{j-1}\right]$$

$$= \left(\lambda_j^2 + b^2h^2 \sin^2 j\pi h\right)x_j + (bh \sin j\pi h)\left[1, 0, \dots, 0, (-1)^j\right]$$

$$-\left(\frac{b^2h^2}{2} \sin j\pi h\right)\left[1, 0, \dots, 0, (-1)^{j-1}\right].$$

Thus the result follows on noting that $\sin ni\pi h = (-1)^{i-1} \sin i\pi h$.

LEMMA 2.3.

$$\|x_j\|^2 = \sum_{k=1}^{n} \sin^2 jk\pi h = \frac{n+1}{2}$$

*and*

$$\sum_{k \text{ odd}} \sin^2 jk\pi h \sim \frac{n+1}{4}.$$

COROLLARY.

$$\sum_i c_i \sin i\pi h \leqslant \left(\sum c_i^2\right)^{1/2}\left(\sum \sin^2 i\pi h\right)^{1/2} = \left(\sum c_i^2\right)^{1/2}\sqrt{\frac{n+1}{2}}.$$

*Proof of Lemma 2.3.*

$$\begin{aligned}
\sum_{k=1}^{n} \sin^2 jk\pi h &= \sum_{k=1}^{n} \frac{1-\cos(2jk\pi h)}{2} \\
&= \frac{n}{2} - \frac{1}{2}\sum_{k=1}^{n} \cos(2jk\pi h) \\
&= \frac{n}{2} - \frac{1}{2}\operatorname{Re}\sum_{k=1}^{n} e^{ik\theta} = \frac{n}{2} - \frac{1}{2}\operatorname{Re}\frac{z(1-z^n)}{1-z},
\end{aligned}$$

where $\theta = 2j\pi h$. Now we compute $\operatorname{Re}[z(1-z^n)/(1-z)]$:

$$\begin{aligned}
\operatorname{Re}\frac{z(1-z^n)}{1-z} &= \operatorname{Re}\frac{z(1-\bar{z})(1-z^n)}{(1-z)(1-\bar{z})} = -\operatorname{Re}\frac{(1-z-z^n+z^{n+1})}{(1-z)(1-\bar{z})} \\
&= \frac{-(1-\cos\theta)+\cos n\theta-\cos(n+1)\theta}{2(1-\cos\theta)} \\
&= -\frac{1}{2} - \frac{2\sin\left(\dfrac{2n+1}{2}\theta\right)\sin\left(-\dfrac{\theta}{2}\right)}{2(1-\cos\theta)}.
\end{aligned}$$

Noting that

$$\sin\left(\frac{2n+1}{2}\theta\right) = \sin\left(-\frac{\theta}{2}\right),$$

the first estimate follows. The second estimate is now obvious. ∎

REMARK 2.1. $\nu_j^2 = \lambda_j^2 + b^2h^2\sin^2 j\pi h$ can be regarded as approximate eigenvalues of $J_B^T J_B$, since

$$\frac{\|x_j^T J_B^T J_B x_j\|}{\|x_j\|^2} = \lambda_j^2 + b^2h^2\sin^2 j\pi h - \frac{b^2h^2\sin^2 j\pi h}{\|x_j\|^2}.$$

THEOREM 2.4. *Assume* $\sigma(J) \subset [d_0^2h^2, d_1^2] = [\lambda_1, \lambda_n]$ *and* $s = 1/d_0d_1h$. *Then*

$$\left\|(I+sJ_B)^{-1}(I-sJ_B)x\right\| \leqslant r_H\|x\|, \tag{2.11}$$

*where*

$$r_H^2 = r_H^2(n) = \frac{\tilde{b} + (1-s\lambda_n)^2}{\tilde{b} + (1+s\lambda_n)^2} \quad \textit{and} \quad \tilde{b} = \frac{|b|(2+|b|h)}{d_0^2d_1^2h}.$$

*Here the choice of* $1/s$ *as the geometric mean of* $\lambda_1$ *and* $\lambda_n$, *as in the symmetric case, is the best possible.*

*Proof.* Let $v_j = x_j/\|x_j\|$ be the normalized eigenvectors of $J$, and put $y = (I+sJ_B)^{-1}x = \Sigma c_jv_j$. Without loss of generality, we may assume $y$ is normalized so that $\Sigma c_i^2 = 1$.

Noting that $(I-sJ_B)(I+sJ_B) = (I+sJ_B)(I-sJ_B)$, we see that (2.11) is equivalent to

$$\left\|(I-sJ_B)y\right\|^2 \leqslant r_H^2\left\|(I+sJ_B)y\right\|^2. \tag{2.12}$$

Expanding, we have

$$y^T\left(I - sJ_B^T - sJ_B + s^2J_B^TJ_B\right)y \leqslant r_H^2\, y^T\left(I + sJ_B^T + sJ_B + s^2J_B^TJ_B\right)y,$$

or

$$(1 - r_H^2)(y^T y + s^2 y^T J_B^T J_B y) \leqslant s(1 + r_H^2) y^T (J_B^T + J_B) y$$
$$= 2s(1 + r_H^2) \sum_i c_i^2 \lambda_i^2. \tag{2.13}$$

But by Lemmas 2.1, 2.2, 2.3 and the Cauchy-Schwarz inequality,

$$y^T J_B^T J_B y = y^T (J^2 + JB - BJ - B^2) y$$
$$= \sum c_i^2 \nu_i^2 + bh \sum_{i,j} \frac{\left[1 + (-1)^{i+j-1}\right] c_i c_j \sin i\pi h \sin j\pi h}{\|x_i\| \cdot \|x_j\|} + O(h^2)$$
$$\leqslant \sum c_i^2 \nu_i^2 + 2|b|h \leqslant \sum c_i^2 \lambda_i^2 + b^2 h^2 + 2|b|h.$$

Thus, it suffices to show

$$(1 - r_H^2)\left[1 + s^2\left(\sum c_i^2 \lambda_i^2 + b^2 h^2 + 2|b|h\right)\right] \leqslant 2s(1 + r_H^2) \sum c_i^2 \lambda_i^2,$$

or

$$\sum c_i^2 (1 - s\lambda_i)^2 + \tilde{b} \leqslant r_H^2 \left[\sum c_i^2 (1 + s\lambda_i)^2 + \tilde{b}\right]. \tag{2.14}$$

When there are no convective terms, it is well known [2, 3, 5] that the optimal choice of $1/s$ is the geometric mean of two extreme eigenvalues to minimize $r_H^2$. But when $b \neq 0$, we do not have an optimal choice *a priori*. Instead, let $1/s$ be any number in $\sigma(J)$, unspecified for the time being, and let $n_0$ be the index for which $\max_i [(1 - s\lambda_i)/(1 + s\lambda_i)]^2$ is attained, and let $r_H^2 = r_H^2(n_0)$. Let

$$\bar{r}^2 = \left(\frac{1 - s\lambda_{n_0}}{1 + s\lambda_{n_0}}\right)^2 = \max_i \left(\frac{1 - s\lambda_i}{1 + s\lambda_i}\right)^2.$$

Then $\bar{r}^2 < r_H^2(n_0)$, since $\tilde{b} > 0$. Then the left hand side of (2.14) is less than

$$\begin{aligned}\bar{r}^2 \sum c_i^2(1 + s\lambda_i)^2 + \tilde{b} &= r_H^2\left(\sum c_i^2(1 + s\lambda_i)^2 + \tilde{b}\right)\\ &\quad + \left(\bar{r}^2 - r_H^2\right)\sum c_i^2(1 + s\lambda_i)^2 + \left(1 - r_H^2\right)\tilde{b}\\ &\leqslant r_H^2\left(\sum c_i^2(1 + s\lambda_i)^2 + \tilde{b}\right)\\ &\quad + \left(\bar{r}^2 - r_H^2\right)(1 + s\lambda_n)^2 + \left(1 - r_H^2\right)\tilde{b}.\end{aligned}$$

We claim that the sum of the last two terms is less than or equal to zero. We have

$$\begin{aligned}\bar{r}^2 - r_H^2 &= \left(\frac{1 - s\lambda_{n_0}}{1 + s\lambda_{n_0}}\right)^2 - \frac{\tilde{b} + \left(1 - s\lambda_{n_0}\right)^2}{\tilde{b} + \left(1 + s\lambda_{n_0}\right)^2}\\ &= \frac{-4s\lambda_{n_0}\tilde{b}}{\left(1 + s\lambda_{n_0}\right)^2\left[\tilde{b} + \left(1 + s\lambda_{n_0}\right)^2\right]}.\end{aligned}$$

But

$$\left(1 - r_H^2\right)\tilde{b} = \frac{4s\lambda_{n_0}\tilde{b}}{\tilde{b} + \left(1 + s\lambda_{n_0}\right)^2}.$$

Thus, the sum of the last two terms is less than or equal to zero because $\lambda_n \geqslant \lambda_{n_0}$. This proves (2.14).

Now we discuss the optimal choice of $s$. Let $s_0 = 1/d_0 d_1 h = 1/\sqrt{\lambda_1 \lambda_n}$ be the optimal choice for the symmetric case. First assume $s < s_0$. Then the maximum of $[(1 - s\lambda_i)/(1 + s\lambda_i)]^2$ is attained at $\lambda_1$ $(n_0 = 1)$. In this case we claim that $r_H^2(n)$ is better (smaller) than $r_H^2(1)$. Indeed, we compute

$$\frac{\tilde{b} + (1 - s\lambda_1)^2}{\tilde{b} + (1 + s\lambda_1)^2} - \frac{\tilde{b} + (1 - s_0\lambda_n)^2}{\tilde{b} + (1 + s_0\lambda_n)^2}. \tag{2.15}$$

Its numerator is

$$4(s_0\lambda_n - s\lambda_1)\left(\tilde{b} + 1 - ss_0\lambda_n\lambda_1\right). \tag{2.16}$$

The first factor $(s_0\lambda_n - s\lambda_1)$ is obviously positive, and $\tilde{b} + 1 - ss_0\lambda_1$ is positive because $s_0\lambda_n = \sqrt{\lambda_n/\lambda_1}$, $s\lambda_1 < s_0\lambda_1 = \sqrt{\lambda_1/\lambda_n}$. Thus $s_0$ is a better choice, resulting in the convergence factor $r_H^2 = r_H^2(n)$. Now assume $s > s_0$. In this case the maximum is attained at $\lambda_n$ $(n_0 = n)$. Now the first factor of (2.16) ($\lambda_n$ in place of $\lambda_1$), $(s_0 - s)\lambda_n$, is negative, while the second factor $\tilde{b} + 1 - ss_0\lambda_n\lambda_n$ is negative also asymptotically because $\tilde{b} + 1 = O(h^{-1})$ and $ss_0\lambda_n\lambda_n > s_0^2\lambda_n^2 > O(h^{-2})$. In either case, we conclude that $s_0$ is the best choice and $r_H^2 = r_H^2(n)$. ∎

THEOREM 2.5. *Assume* $\sigma(J) \subset [d_0^2h^2, d_1^2]$ *and* $s = 1/d_0d_1h$. *Then*

$$\left\|(I - sV_Q)(I + sV_Q)^{-1}x\right\| \leqslant r_V\|x\|, \tag{2.17}$$

*where* $r_V^2$ *is similar to that of Theorem* 2.4, *with* $c$ *replacing* $b$.

*Proof.* Since $V_Q$ has the same block structure as $H_P$ on reordering the nodes, the proof is the same as for Theorem 2.4. ∎

Now we have the following:

THEOREM 2.6. *The convergence factor of the ADI iteration in the nonsymmetric case is asymptotically the same as the symmetric case, i.e.,*

$$\begin{aligned}\rho(G_s) &= \rho(\tilde{G}_s) \leqslant \|\tilde{G}_s\|\\ &\leqslant \left\|(I + sH_P)^{-1}(I - sH_P)\right\|\cdot\left\|(I - sV_Q)(I + SV_Q)^{-1}\right\|\\ &\leqslant r_Hr_V = \frac{1 - Ch}{1 + Ch},\end{aligned}$$

*where the constant* $C$ *is the same as that of the symmetric case. Furthermore, the convergence is unaffected by the presence of nonsymmetric terms.*

*Proof.* Write $\tilde{b} = 2\bar{b}/h$. Then one can easily see that

$$r_H^2 = \frac{1 + \dfrac{2(\bar{b} - D_1)h}{D_1^2} + \dfrac{h^2}{D_1^2}}{1 + \dfrac{2(\bar{b} + D_1)h}{D_1^2} + \dfrac{h^2}{D_1^2}}, \tag{2.18}$$

where $D_1 = d_1/d_0$. If $\bar{b} = 0$, this reduces to the symmetric case

$$\frac{1 - \dfrac{2h}{D_1} + \dfrac{h^2}{D_1^2}}{1 + \dfrac{2h}{D_1} + \dfrac{h^2}{D_1^2}}. \tag{2.19}$$

We subtract (2.19) from (2.18), neglecting second order terms, to get

$$\frac{4\bar{b}h^2}{D_1^3[1 + O(h)]}.$$

This is second order in $h$, which means that, asymptotically, the convergence factor is as good as in the symmetric case. The estimate for $r_V^2$ follows exactly the same way. As a consequence the second assertion follows. ■

## 3. NUMERICAL EXAMPLE

We solve

$$-\Delta u + \vec{b} \cdot \nabla u = f \qquad \text{in } \Omega,$$

$$u = g \qquad \text{on } \partial\Omega,$$

with $\vec{b} = (b, c)$, and $\Omega$ the unit square. We stop when the residual error is less than $10^{-6}$. The numerical experiment (Table 1) shows that the number

TABLE 1

| | Number of Iterations | | | |
|---|---|---|---|---|
| $h$ | $b = c = 5.0$ | 10.0 | 30.0 | 50.0 |
| $\frac{1}{5}$ | 7 | 9 | NA | NA |
| $\frac{1}{10}$ | 13 | 11 | *21 | NA |
| $\frac{1}{15}$ | 19 | 17 | 16 | NA |
| $\frac{1}{20}$ | 25 | 22 | 17 | *27 |
| $\frac{1}{25}$ | 31 | 27 | 22 | 22 |
| $\frac{1}{30}$ | 37 | 31 | 30 | 20 |
| $\frac{1}{50}$ | 60 | 50 | 45 | 39 |

of iterations grows linearly with $\sqrt{N} = 1/h$, which is in agreement with the theory. The entries with $*$ show that the algorithm converges when both $bh/2$ and $ch/2$ are greater than 1. When $b$ or $c$ becomes larger, the solution is not available for some $h$. This is due to the indefiniteness of the matrix $A$. If convective terms are large, the discretization only makes sense when $h$ is sufficiently small.

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